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ADAPTIVE RELIABILITY ESTIMATION UNDER PROGRESSIVE CENSORING FOR INDUSTRIAL DECISION SUPPORT AND PREDICTIVE MAINTENANCE APPLICATIONS

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Original research

A B S T R A C T

A risk-improved data-adaptive shrinkage estimator is proposed for the reliability function of the exponentiated Weibull distribution under progressive Type-II censoring, with emphasis on both finite-sample risk reduction and decision support in industrial reliability studies. The exponentiated Weibull maximum likelihood plug-in estimator is combined with a structured Weibull-target reliability estimator through a time-dependent shrinkage weight. The weight is determined by an observed-information risk rule that balances the estimated variability of the full-model estimator against the squared discrepancy between the full-model and target reliability curves. A Monte Carlo study over representative sample sizes and censoring proportions shows that the proposed estimator improves the full exponentiated Weibull reliability estimator in integrated mean squared error, with the largest gains appearing under heavier censoring. The proposed reliability curve is then interpreted as a practical input for threshold-based maintenance timing, inspection scheduling, warranty assessment, and risk-based operational planning. In this way, the method contributes not only a statistically coherent adaptive shrinkage mechanism, but also a more stable reliability tool for censored industrial life-testing data.

1. INTRODUCTION

Reliability analysis is a central component of statistical decision-making for lifetime data in engineering, industrial experiments, medicine, and survival studies (Liu etal., 2025; Meeker & Hong, 2014; Yazdi, 2024). In many practical settings, complete failure times are not observed because experiments are terminated early, surviving units are withdrawn, or long tests are too costly to continue (Alliu et al., 2025; Kumari et al., 2024). Progressive Type-II censoring is therefore important because it preserves useful lifetime information while reflecting the realities of time-limited and resource-limited testing programs discussed by Cohen (1963), Balakrishnan and Aggarwala (2000), Balakrishnan and Cramer (2014) and Tian et al. (2024).

The exponentiated Weibull distribution is a flexible lifetime model that extends the classical Weibull distribution by adding an additional shape parameter (Gómez et al., 2023; Zanboori et al., 2025). This flexibility allows the density and hazard functions to exhibit increasing, decreasing, unimodal, and bathtub-type patterns, which explains its continued use in reliability studies after the work of Mudholkar and Srivastava (1993) and Mudholkar et al. (1995). At the same time, the additional flexibility may create unstable estimation when the effective sample information is reduced by small sample size or heavy censoring (Riley et al., 2025; Ghorbal, 2026).

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Maximum likelihood estimation remains the standard starting point for censored lifetime models, but a highly flexible model does not automatically produce the most stable reliability curve in finite samples. For the exponentiated Weibull distribution, the likelihood equations are nonlinear and the plug-in reliability estimator may display substantial variance under limited information. This motivates shrinkage estimation, where a data-driven estimator is stabilized by borrowing strength from a structured target; see Stein (1956), James and Stein (1961), and Casella (1985).

The practical importance of this issue is not purely statistical. Estimated reliability curves are often used to determine inspection intervals, preventive replacement horizons, warranty exposure, and acceptable operating windows. As emphasized in classical reliability references such as Nelson (1982), Lawless (2003), and Meeker and Escobar (1998), unstable reliability estimation can lead directly to unstable maintenance and quality decisions. A method that reduces curve-level estimation noise under censoring is therefore valuable not only for inference, but also for operational planning.

This paper develops a risk-improved data-adaptive shrinkage estimator for the reliability function of the exponentiated Weibull distribution under progressive Type-II censoring. The proposed estimator shrinks the exponentiated Weibull maximum likelihood reliability estimator toward a structured target obtained from the Weibull submodel. The shrinkage weight is not fixed; instead, it is estimated as a time-dependent function through an observed-information risk rule. The comparison is performed for the same statistical quantity, namely the reliability curve R(t), over quantile-based grids rather than at a single arbitrarily chosen mission time. This formulation makes the method mathematically coherent and also suitable for decision-oriented reliability interpretation.

The main contributions of this paper are summarized as follows:

1. A structured Weibull-target reliability estimator is constructed within the exponentiated Weibull family.
2. A new data-adaptive shrinkage reliability estimator is proposed for progressive Type-II censored samples.
3. The shrinkage weight is allowed to depend on time and is estimated from the data through an observed-information risk rule.
4. The comparison is performed for the reliability curve R(t), avoiding comparisons among different statistical quantities.
5. Integrated bias, integrated mean squared error, and relative efficiency are used to evaluate the estimators over quantile-based reliability grids.
6. The reliability curve is further interpreted for threshold-based maintenance, inspection, and risk decisions using the same estimator developed in the statistical model.

2. EXPONENTIATED WEIBULL RELIABILITY MODEL

Let X be a nonnegative lifetime random variable following the exponentiated Weibull distribution with parameter vector

$$\theta = (\alpha, \beta, \eta)^T, \qquad \alpha > 0, \quad \beta > 0, \quad \eta > 0.$$

The cumulative distribution function is given by

$$F(x; \alpha, \beta, \eta) = \left(1 - \exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right]\right)^{\alpha}, \qquad x > 0.$$

The corresponding reliability function is

$$R(x; \alpha, \beta, \eta) = 1 - \left(1 - \exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right]\right)^{\alpha}.$$

The probability density function is

$$f(x; \alpha, \beta, \eta) = \frac{\alpha\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right]\left(1 - \exp\left[-\left(\frac{x}{\eta}\right)^{\beta}\right]\right)^{\alpha-1}.$$

The hazard rate function is defined by

$$h(x; \alpha, \beta, \eta) = \frac{f(x; \alpha, \beta, \eta)}{R(x; \alpha, \beta, \eta)}.$$

The quantile function is useful in simulation and in defining reliability grids. If $0 < p < 1$, then

$$Q(p; \alpha, \beta, \eta) = \eta\left[-\log\left(1 - p^{1/\alpha}\right)\right]^{1/\beta}.$$

When $\alpha = 1$, the exponentiated Weibull distribution reduces to the ordinary Weibull distribution. This submodel will be used later to construct a structured shrinkage target.

2.1 Progressive Type-II Censoring

Suppose that n identical units are placed on a life test. The experiment continues until m failures are observed, where $m < n$. Let

$$x_{1:m:n} < x_{2:m:n} < \cdots < x_{m:m:n}$$

denote the observed ordered failure times. At the time of the i-th failure, r_i surviving units are randomly withdrawn from the experiment. The censoring scheme is denoted by

$$\mathbf{r} = (r_1, r_2, \ldots, r_m), \qquad \sum_{i=1}^{m} r_i + m = n.$$

For simplicity, we write $x_i = x_{i:m:n}$. Under progressive Type-II censoring, the likelihood function is

$$L(\theta) = C \prod_{i=1}^{m} f(x_i; \theta)[R(x_i; \theta)]^{r_i},$$

Where:

$$C = n(n - r_1 - 1)(n - r_1 - r_2 - 2) \cdots (n - r_1 - \cdots - r_{m-1} - m + 1)$$

is a constant independent of the model parameters.

Using the exponentiated Weibull density and reliability functions in [eq:ew_pdf] and [eq:ew_reliability], the log-likelihood function, apart from an additive constant, is

$$\ell(\alpha,\beta,\eta) = m\log\alpha + m\log\beta - m\log\eta + (\beta-1)\sum_{i=1}^{m}\log\left(\frac{x_i}{\eta}\right) - \sum_{i=1}^{m} z_i + (\alpha-1)\sum_{i=1}^{m}\log(1-e^{-z_i}) + \sum_{i=1}^{m} r_i \log[1-(1-e^{-z_i})^{\alpha}],$$

where

$$z_i = \left(\frac{x_i}{\eta}\right)^{\beta}.$$

2.2 Maximum Likelihood Reliability Estimation

The maximum likelihood estimator of $\theta = (\alpha,\beta,\eta)^T$ is defined as

$$\hat{\theta}_{EW} = \left(\hat{\alpha},\hat{\beta},\hat{\eta}\right)^T = \arg\max_{\alpha,\beta,\eta>0}\ell(\alpha,\beta,\eta).$$

Since the likelihood equations are nonlinear, the estimator is obtained by numerical maximization of [eq:ew_loglik]. The plug-in maximum likelihood estimator of the reliability function is then

$$\hat{R}_{EW}(t) = R\left(t;\hat{\alpha},\hat{\beta},\hat{\eta}\right).$$

This estimator uses the full exponentiated Weibull model. It is flexible, but it may have relatively high variability when the sample size is small or when the censoring level is high.

2.3 Weibull-Target Reliability Estimation

The ordinary Weibull model is obtained from the exponentiated Weibull distribution by setting $\alpha = 1$. In this case, the reliability function becomes

$$R_W(t;\beta,\eta) = \exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right].$$

The corresponding density is

$$f_W(t;\beta,\eta) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}\exp\left[-\left(\frac{t}{\eta}\right)^{\beta}\right].$$

Under the same progressive Type-II censoring scheme, the Weibull log-likelihood, apart from a constant, is

$$\ell_W(\beta,\eta) = m\log\beta - m\log\eta + (\beta-1)\sum_{i=1}^{m}\log\left(\frac{x_i}{\eta}\right) - \sum_{i=1}^{m}(1+r_i)\left(\frac{x_i}{\eta}\right)^{\beta}.$$

Let

$$\hat{\phi}_W = \left(\hat{\beta}_W,\hat{\eta}_W\right)^T = \arg\max_{\beta,\eta>0}\ell_W(\beta,\eta).$$

The Weibull-target reliability estimator is defined as

$$\hat{R}_W(t) = R_W\left(t;\hat{\beta}_W,\hat{\eta}_W\right).$$

The Weibull-target estimator is not introduced as a competing model for all situations. Instead, it is used as a structured and stable target inside the exponentiated Weibull family. When the full exponentiated Weibull fit is unstable, the Weibull target can provide useful stabilization. When the Weibull target is far from the full-model fit, the proposed data-adaptive weight will reduce the amount of shrinkage.

2.4 Proposed Data-Adaptive Shrinkage Reliability Estimator

The main quantity of interest in this paper is the reliability function $R(t)$. Therefore, the proposed shrinkage is constructed directly at the reliability-function level, not merely at the parameter level. This avoids comparing different statistical quantities and ensures that all estimators are evaluated for the same target curve.

Let $\hat{R}_{EW}(t)$ be the full-model plug-in reliability estimator in [eq:rew_mle], and let $\hat{R}_W(t)$ be the Weibull-target reliability estimator in [eq:rw_target]. The proposed data-adaptive shrinkage reliability estimator is defined by

$$\hat{R}_{ASH}(t) = \hat{a}(t)\hat{R}_W(t) + [1-\hat{a}(t)]\hat{R}_{EW}(t), \qquad 0 \le \hat{a}(t) \le 1.$$

Here, $\hat{a}(t)$ is a time-dependent shrinkage weight estimated from the data. A larger value of $\hat{a}(t)$ gives more weight to the Weibull target, while a smaller value gives more weight to the full exponentiated Weibull estimator.

The proposed weight is motivated by the bias-variance trade-off. Let \widehat V_{EW}(t) denote an observed-information estimate of the variability of \hat R_{EW}(t). Let

$$\hat{D}^2(t) = \left[\hat{R}_{EW}(t) - \hat{R}_W(t)\right]^2$$

measure the squared discrepancy between the full-model estimator and the target estimator. The proposed shrinkage weight is

$$\hat{a}(t) = \frac{\hat{V}_{EW}(t)}{\hat{V}_{EW}(t) + \hat{D}^2(t) + \varepsilon},$$

where $\varepsilon > 0$ is a small numerical constant used to avoid division by zero.

The interpretation of [eq:adaptive_weight] is clear. If the full exponentiated Weibull reliability estimator has high variability and the Weibull target is close to it, then $\hat{a}(t)$ becomes large and stronger shrinkage is applied. If the Weibull target is far from the full-model estimator, then $\hat{D}^2(t)$ becomes large and the amount of shrinkage is automatically reduced. Thus, the estimator avoids a fixed shrinkage constant and allows the data to determine the appropriate amount of shrinkage at each time point.

Observed-Information Estimation of the Weight

An observed-information approximation is used to estimate \widehat V_{EW}(t). The procedure is as follows:

1. Fit the exponentiated Weibull model to the progressive Type-II censored data and obtain \hat\theta_{EW}.
2. Compute the full-model reliability estimate \hat R_{EW}(t) on the chosen reliability grid.

3. Evaluate the observed information matrix at the exponentiated Weibull maximum likelihood estimate.
4. Estimate the reliability variance by a delta-method approximation based on the inverse observed information matrix.

$$\hat{V}_{EW}(t) = \frac{1}{B-1}\sum_{b=1}^{B}\left[\hat{R}_{EW}^{*(b)}(t) - \bar{R}_{EW}^{*}(t)\right]^2,$$

1. Compute the discrepancy between the full-model and Weibull-target reliability estimators.

$$\bar{R}_{EW}^{*}(t) = \frac{1}{B}\sum_{b=1}^{B}\hat{R}_{EW}^{*(b)}(t).$$

2. Compute \hat a(t) using Eq. (17).
3. Obtain the proposed estimator \hat R_{ASH}(t) using Eq. (16).

This method makes both the shrinkage amount and the reliability evaluation grid data-dependent. It therefore avoids the use of arbitrary fixed mission times and fixed shrinkage constants.

2.5 Reliability Grids and Performance Measures

In the simulation study, the reliability curve is evaluated over a quantile-based grid. For a parameter setting θ, define

$$\mathcal{T}_\theta = \{Q(p_j;\theta):\ p_j = 0.10, 0.20, \ldots, 0.90\}.$$

This grid represents different regions of the lifetime distribution and avoids the arbitrary choice of a single fixed time point.

For the Monte Carlo study, the grid is defined from the true exponentiated Weibull quantiles in order to compare all estimators on the same reliability scale.

$$\hat{\mathcal{T}} = \{Q(p_j;\hat{\theta}_{EW}):\ p_j = 0.10, 0.20, \ldots, 0.90\}.$$

Let $\hat{R}_s^{(j)}(t)$ denote the reliability estimate obtained by method s in the j-th Monte Carlo replication, where $s \in \{EW, W, ASH\}$. The integrated bias is defined by

$$\mathrm{IBias}_s = \frac{1}{q}\sum_{\ell=1}^{q}\left[\frac{1}{M}\sum_{j=1}^{M}\hat{R}_s^{(j)}(t_\ell) - R(t_\ell)\right],$$

where q is the number of grid points. The integrated mean squared error is defined by

$$\mathrm{IMSE}_s = \frac{1}{q}\sum_{\ell=1}^{q}\frac{1}{M}\sum_{j=1}^{M}\left[\hat{R}_s^{(j)}(t_\ell) - R(t_\ell)\right]^2.$$

The relative efficiency of the proposed estimator with respect to the full-model plug-in estimator is

$$\mathrm{RE}_{ASH} = \frac{\mathrm{IMSE}_{EW}}{\mathrm{IMSE}_{ASH}}.$$

Thus, $\mathrm{RE}_{ASH} > 1$ indicates that the proposed data-adaptive shrinkage estimator has a smaller integrated mean squared error than the full exponentiated Weibull maximum likelihood reliability estimator.

Monte Carlo Simulation Study

A Monte Carlo simulation study was conducted to evaluate the finite-sample behavior of the proposed estimator. Lifetime observations were generated from the exponentiated Weibull distribution using the quantile representation

$$X = \eta\left[-\log\left(1 - U^{1/\alpha}\right)\right]^{1/\beta}, \qquad U \sim U(0,1).$$

After generating a complete sample of size n, the progressive Type-II censoring mechanism was applied according to a pre-specified censoring scheme r.

The simulation focuses on the representative parameter setting

$$\theta_1 = (1.5, 2.0, 1.0)^T, \qquad \theta_2 = (0.8, 1.5, 1.0)^T, \qquad \theta_3 = (2.0, 0.8, 1.0)^T.$$

with sample sizes

$$n = 30,\ 50,\ 100,$$

and censoring proportions

$$20\%,\ 40\%,\ 60\%.$$

For each combination of sample size and censoring proportion, M=50 Monte Carlo replications were used. The adaptive weight was computed from an observed-information variance estimate.

2.6 Monte Carlo simulation design

Monte Carlo simulation design and experimental settings for evaluating the reliability curve estimators is given in Table 1.

Table 1. Monte Carlo Simulation Design and Experimental Settings for Evaluating the Reliability Curve Estimators

Item	Values
Distribution	Exponentiated Weibull
Parameter setting	Reported tables: θ1=(1.5, 2.0, 1.0)^T
Sample sizes	n = 30, 50, 100
Censoring proportions	20%, 40%, 60%
Monte Carlo replications	M = 50
Weight estimation	Observed-information variance + discrepancy rule
Reliability grid	Q(0.10), Q(0.20), ..., Q(0.90)
Compared estimators	R̂_EW(t), R̂_W(t), R̂_ASH(t)
Performance measures	IBias, IMSE, RE

The censoring scheme distributes removals approximately evenly over the observed failure times. This provides a stable benchmark for comparing the exponentiated Weibull, Weibull-target, and adaptive shrinkage reliability estimators.

Table 2 reports the integrated bias and integrated mean squared error for the three competing reliability estimators under the representative setting $\theta_1=(1.5, 2.0, 1.0)^T$. The proposed adaptive shrinkage estimator improves the exponentiated Weibull benchmark in every reported scenario, but the magnitude of improvement depends on the information loss induced by censoring.

Table 2. Integrated bias and integrated mean squared error

n	Censoring	IBias(EW)	IMSE(EW)	IBias(W)	IMSE(W)	IBias(ASH)	IMSE(ASH)
30	20%	0.0032	0.0050	0.0078	0.0048	0.0069	0.0048
30	40%	-0.0040	0.0198	-0.0031	0.0062	-0.0014	0.0192
30	60%	-0.0238	0.0149	-0.0129	0.0104	-0.0219	0.0140
50	20%	0.0092	0.0089	0.0009	0.0025	0.0113	0.0088
50	40%	-0.0174	0.0068	-0.0172	0.0032	-0.0147	0.0063
50	60%	-0.0287	0.0093	-0.0191	0.0063	-0.0280	0.0085
100	20%	0.0038	0.0012	0.0064	0.0012	0.0062	0.0012
100	40%	-0.0002	0.0018	0.0035	0.0015	0.0027	0.0016
100	60%	-0.0085	0.0055	-0.0095	0.0029	-0.0079	0.0050

Table 3. Relative efficiency of the proposed estimator

n	Censoring	RE_ASH	Mean weight	Conclusion
30	20%	1.036	0.901	Improved
30	40%	1.033	0.781	Improved
30	60%	1.066	0.823	Improved
50	20%	1.012	0.907	Near tie
50	40%	1.069	0.839	Improved
50	60%	1.095	0.836	Improved
100	20%	1.003	0.921	Near tie
100	40%	1.110	0.896	Improved
100	60%	1.090	0.835	Improved

Table 3 reports the relative efficiency of the proposed estimator relative to the full exponentiated Weibull estimator together with the average adaptive weight. The gains are clearest under moderate and heavy censoring. At n=30, the IMSE decreases from 0.0198 to 0.0192 under 40% censoring and from 0.0149 to 0.0140 under 60% censoring. At n=50, the IMSE falls from 0.0068 to 0.0063 under 40% censoring and from 0.0093 to 0.0085 under 60% censoring. At n=100, the strongest representative gain occurs under 40% censoring, where RE_ASH reaches 1.110.

The numerical results indicate that the proposed adaptive shrinkage estimator improves the exponentiated Weibull plug-in estimator across all reported scenarios. Under light censoring, the relative efficiency is close to one, which is desirable because the full-model estimator is already fairly stable. Under heavier censoring, however, the shrinkage mechanism yields more noticeable protection against finite-sample variability. This pattern is consistent with the design goal of improving reliability estimation precisely in the settings where censoring removes a substantial fraction of the available failure information.

3. INDUSTRIAL RELEVANCE AND RELIABILITY-BASED DECISION SUPPORT

In industrial reliability management, the estimated survival curve is not an end in itself; it is used to support preventive maintenance, inspection planning, warranty evaluation, and asset-risk control. Nelson (1982), Lawless (2003), and Meeker and Escobar (1998) make clear that decision quality depends on the stability of the estimated reliability function over time. When a life test is progressively censored, incomplete failure information can make a flexible full-model estimator too variable for direct operational use, especially when plant managers need a smooth and interpretable curve rather than a sequence of unstable point estimates.

A practical way to translate the proposed estimator into a maintenance rule is to define a minimum acceptable reliability level c and then determine the latest operating time at which the fitted curve still satisfies that requirement. Using the proposed adaptive estimator, this decision horizon can be written as:

$$t\_c = \sup\{ t : \hat{R}\_ASH(t) \geq c \}$$

The quantity t_c may be interpreted as a decision-oriented intervention time: before this point the component remains above the required reliability threshold, whereas after this point an inspection, preventive replacement, or intensified monitoring action becomes reasonable. Because the proposed estimator reduces integrated mean squared error relative to the full exponentiated Weibull estimator in the reported designs, the corresponding threshold time is expected to fluctuate less from sample to sample. This is important in predictive maintenance because highly variable threshold times produce unstable service intervals and inconsistent resource allocation.

A second useful construct is the estimated safe operating region:

$$A_c = \{ t : \hat{R}_{ASH}(t) \geq c \}$$

which summarizes the time window over which the system can be regarded as operationally acceptable. In quality assurance and warranty analysis, this set provides a direct way to compare whether different production batches or design options meet a required reliability standard over the same horizon. Since the present method shrinks the exponentiated Weibull estimator toward a structured Weibull target only when that target is sufficiently compatible with the data, it can improve decision stability without forcing the analysis to ignore genuine deviations from Weibull behavior.

The relevance of this interpretation becomes stronger when censoring is heavy. In many industrial life tests, not all units can be observed until failure because time, cost, spare-part availability, or testing capacity force early withdrawals. Progressive Type-II censoring is therefore a realistic abstraction of durability studies in which some units are removed for economic reasons while inference must still be made about the reliability of the broader system. In such cases, a modest reduction in curve-level risk can translate into more stable decisions on inspection timing, replacement prioritization, and reserve inventory planning.

The representative simulation results support exactly this interpretation. The proposed estimator yields RE_{ASH} greater than one throughout the reported scenarios, with the gains becoming more visible at 40% and 60% censoring. This means that the curve used for operational decisions is less noisy precisely in the conditions where practical testing programs are shortest and data limitations are most severe. At the same time, the method remains conservative under light censoring and large sample sizes, where the full exponentiated Weibull estimator is already stable and adaptive shrinkage should not distort the original signal. From an industrial viewpoint, this balance between flexibility and stability is one of the main strengths of the proposed framework.

Although the proposed methodology is evaluated through Monte Carlo simulation, its intended application is motivated by practical reliability problems encountered in industrial environments. In predictive maintenance programs, lifetime information is frequently incomplete because equipment is replaced preventively, inspections are performed before failure, or testing campaigns are terminated due to operational constraints. These situations naturally generate progressively censored observations similar to those considered in this study. The proposed adaptive reliability estimator can therefore be incorporated into existing reliability analysis workflows to support maintenance scheduling, inspection planning, warranty assessment, and operational risk evaluation. Future research will focus on validating the proposed framework using real industrial life-testing data from manufacturing and maintenance systems.

4. DISCUSSION

The numerical study shows that the proposed estimator consistently reduces integrated mean squared error relative to the full exponentiated Weibull estimator in the representative design. The gains are modest under light censoring and become more noticeable when censoring is heavier, which is the regime in which the flexible three-parameter model has the greatest difficulty maintaining stable curve estimation.

The observed-information weight remains close to one when the Weibull target is close to the full-model fit, but it decreases as censoring intensifies and the discrepancy term becomes more influential. This behavior prevents excessive shrinkage while preserving stability. From a practical perspective, it also means that the proposed estimator is less likely to generate overly optimistic reliability curves or excessively conservative maintenance thresholds when data are sparse.

In the reported scenarios, the Weibull target itself often has smaller IMSE than the full-model plug-in estimator, which explains why the adaptive shrinkage estimator moves toward the target and still improves the exponentiated Weibull benchmark. Equally important, the relative efficiencies that are close to one under light censoring show that the estimator does not impose unnecessary shrinkage when the full-model estimate is already adequate. This adaptive balance is exactly what is needed for industrial reliability settings where the amount of available information can vary substantially across tests.

5. CONCLUSION

This paper introduced a risk-improved data-adaptive shrinkage estimator for the reliability function of the exponentiated Weibull distribution under progressive Type-II censoring. The estimator is constructed directly at the reliability-function level by shrinking the full exponentiated Weibull plug-in reliability estimator toward a structured Weibull-target reliability estimator.

The shrinkage weight is estimated from the data by an observed-information risk rule that accounts for both the variability of the full-model estimator and the discrepancy between the full model and the target. This allows the amount of shrinkage to vary across time and adapt to the information available in the censored sample.

The Monte Carlo results show that the proposed estimator improves the exponentiated Weibull plug-in estimator in all reported scenarios, with relative efficiency greater than one throughout the representative simulation design. The strongest gains occur when censoring is moderate or heavy, which is precisely the setting in which practical reliability studies often suffer from information loss.

From an applied viewpoint, the proposed estimator can be used to support threshold-based maintenance timing, inspection scheduling, warranty evaluation, and riskinformed operational planning through a more stable estimated reliability curve. Overall, the method provides a useful compromise between the flexibility of the exponentiated Weibull distribution and the stability of the Weibull submodel. It is especially attractive when progressive censoring reduces the effective sample information but operational decisions must still be made from the available life-testing data.

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Conflict of Interest
The authors declare that they have no conflict of interest.

Data Availability
The simulation results reported in this study were generated from the Monte Carlo design described in the manuscript. No external real-data set is used in the present version. The emphasis of the current revision is on the industrial interpretation of the censored reliability curve generated by the proposed estimator.

Author Contributions
Both authors contributed to the conception of the study, methodological development, simulation design, interpretation of results, and preparation of the manuscript. Both authors read and approved the final version of the manuscript.

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